# Morlet's Theorem

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We want to understand a little more deeply the proof of so called Morlets theorem, that is the statement that

$$\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}\frac{\mathrm{Top}_n}{\mathrm{O}_n}$$

The two references we have in our hands are [Kup] and the origial proof [BL]. Throughout we assume that we are not anywhere near dimension 4.

The idea in [BL] is that Alexanders trick implies that $B\mathrm{Diff}_\partial(D^n) \simeq TD^{D^n}_{\partial D^n}$ where the LHS denotes the geometric realisation of some quotient of simplicial sets of the self homotopies against self diffeomorphisms. Next they construct a bundle $P^t$ such that it is clear from the parallalizability of $D^n$ that $\Gamma^{\partial D^n}(P^t) \simeq \Omega^{n+1}\frac{\mathrm{Top}_n}{\mathrm{O}_n}$ and finally they construct a highly connected map

$$TD^{D^n}_{\partial D^n} \to \Gamma^{\partial D^n}(P^t).$$

Kupers on the other hand argues that $B\mathrm{Diff}_\partial(D^n) \simeq \mathrm{Sm}_\partial(D^n)_0$ some set of smooth structures on the disc rel the boundary, while on the other hand $\mathrm{Sm}(\mathbb{R}^n) \simeq \frac{\mathrm{Top}_n}{\mathrm{O}_n}$. He then uses Gromovos h-principle to argue from the functorial properties of Sm that $\mathrm{Sm}(M)$ is given by some space of sections of a certain bundle and then via elementary means one can identify these spaces of sections in this case.

By far the clearest and simplest approach seems to be that in [KS16], which seems to be a better presented version of what Kupers is trying to say, without the unneccessary generality. We therefore follow closely [KS16, Essay 5]. Note that the approaches are not *essentially* different, they are all beating around the same bush, at least it seems to me.

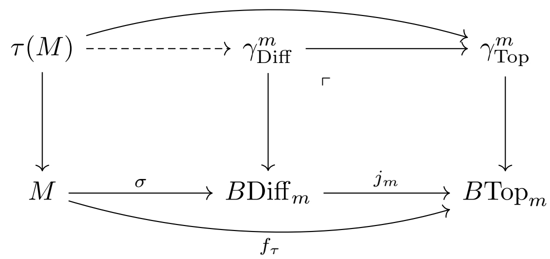
## 1 Generalities

Let $M^m$ be a boundaryless topological manifold, such that at least one smooth structure exists. Let $\mathrm{Diff}(M)$ be the simplicial set whose $k$-simplicies are given by the smooth structures $\Sigma$ on $\Delta^k \times M$ such that the projection $(\Delta^k \times M)_\Sigma \to \Delta^k$ is a submersion.

Let $\tau(M)$ be the tangent microbundle over $M$, I dont know what this is formally. If we assume that the universal bundles for Top and Diff bundles fit into a diagram of the form

$$\begin{array}{ccc} \gamma^m_{\mathrm{Diff}} & \longrightarrow & \gamma^m_{\mathrm{Top}} \\ \downarrow & \lrcorner & \downarrow \\ B\mathrm{Diff}_m & \xrightarrow{j_m} & B\mathrm{Top}_m \end{array}$$

Then we are interested in liftings of the tangent microbundle to a genuine tangent bundle. To make this precise let $f_\tau : M \to B\,\mathrm{Top}_m$ be the classifying map for the tangent microbundle. Then we can define the space of liftings for this bundle (or map) to smooth bundles as $\mathrm{Lift(f_\tau)}$ as a simplicial sets whose $k$-simplicies are given by continuous maps $\sigma : \Delta^k \times M \to B\,\mathrm{Diff}_m$ so that they agree with $f_\tau$, that is $j_m \circ \sigma = f \circ \mathrm{proj}_2 : \Delta^k \times M \to B\,\mathrm{Top}_m$. In particular the zero simplicies are the factorisations of $f$ into a smooth bundle map:




**Lemma** (Classification Theorem 2.3)**.** *There is a homotopy equivalence*

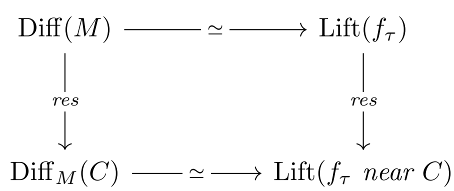
$$\mathrm{Diff}(M) \to \mathrm{Lift(f_\tau)}$$

***Proof.*** (Sketch). They produce a zig zag of homotopy equivalences between intermediate spaces

$$\mathrm{Diff}(M) \to \mathbf{Diff}\,\tau(M) \leftarrow \mathrm{Diff}\,\tau(M) \leftarrow \mathrm{Diff}^{\mathrm{cls}}\,\tau(M) \to \mathrm{Lift}(f_\tau).$$

We will want a relative version of this theorem, to get at which they go through the "near" version. Let $C \subset M$ closed, then $\mathrm{Diff}_M(C)$ is the injective limit over $\mathrm{Diff}(U)$ for $C \subset U \subset M$ open. $\mathrm{Lift}(f_\tau$ near $C)$ is defined likewise as the limit over $\mathrm{Lift}(\mathrm{res}_{\tau(U)}(f_\tau))$ for $C \subset U \subset M$ open.

**Lemma.** *There is a homotopy commutative diagram*




Finally we define the relative groups by taking the fibers of these verticle maps, that is if $a \in \mathrm{Diff}_M(C)$ then $\mathrm{Diff}(M \text{ rel } C_a) := \mathrm{res}^{-1}(a)$. Likewise for liftings. Then because the diagram commutes we get a homotopy equivalence on these fibers:

**Lemma.** *Let $C \subseteq M$ be a closed subset and let $\Sigma$ be a smooth structure on (a neighbourhood of) $C$ and $g$ be the lift of $f_\tau$ near $C$ that corresponds to $\Sigma$ under our map (from the near case), then there is a homotopy equivalence*

$$\mathrm{Diff}(M \text{ rel } C_\Sigma) \simeq \mathrm{Lift}(f_\tau \text{ rel } g_0)$$

This is the theorem that we can plug into to get Morlets theorem. This seeems to be an example of Gromovs h-principle, where Diff is the flexible sheaf and the space of lifts is the sheaf that he associates to it.

## 2 Morlets Theorem

Let $M = \mathbb{R}^n$ and $C = \mathbb{R}^m - D^n_{\mathrm{int}}$ that is $\mathbb{R}^m$ with the interior of the disc removed. Clearly $M$ doesnt have a boundary and $C$ is closed inside of it. We give $C$ the standard structure. Let $f = g_0 = *$ be some constant map Why can I do this....

**Lemma.** $\mathrm{Diff}(\mathbb{R}^n \ rel\ C_\Sigma) = \mathrm{Diff}(D^n \ rel\ \partial D)$.

***Proof.*** First the right hand side needs to be defined, as we have only dealt with the boundaryless case. It is the singular complex associated to the subspace of maps $D^n/\partial D^n \to \mathrm{Top}_n / \mathrm{Diff}_n$ that send a neighbourhood of the base point to the base point. I wouldnt want to prove this properly but it seems entirely intuitive, we have just taken the compliment of the disc. Is that true in general diff of compliment is diff.... □

**Lemma.** $\Omega\, \mathrm{Diff}(D^n \ rel\ \partial D) \simeq \mathrm{Diff}_\partial(D^n)$.

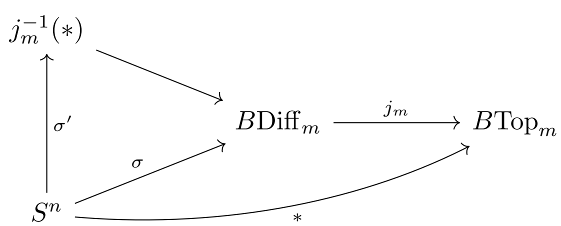
***Proof.*** There is a (Kan) fibration Not entirely clear to me seems like something that is probably in Luck or standard ...

$$\mathrm{Diff}_\partial(D^n) \to \mathrm{Homeo}_\partial(D^n) \to \mathrm{Diff}(D^n \text{ rel } \partial D)$$

Alexanders trick implies that (or simply is that) $\mathrm{Homeo}_\partial(D^n)$ is contractable. Hence by the standard Puppe argument $\Omega\, \mathrm{Diff}(D^n \text{ rel } \partial D) \simeq \mathrm{Diff}_\partial(D^n)$. □

**Lemma.** $\mathrm{Lift}(f_\tau \ rel\ g_0) \simeq \Omega^n\, \mathrm{Top}_n / \mathrm{Diff}_n$

***Proof.*** Without the formalities we can think of the LHS as maps $D^n/\partial D^n = S^n \to B\,\mathrm{Diff}_n$ that factor $f : D^n/\partial D^n = S^n \to B\,\mathrm{Top}_m$. On the other hand the $\mathrm{Top}_n / \mathrm{Diff}_n$ is the fiber of the map $j_m$ over $f$. Because $f$ is constant then to factor it through $j_m$ is exactly to be in the fiber of that constant over $j_m$. Hence $\mathrm{Lift}(* \text{ rel } *) = [S^n, j_m^{-1}(*)] = \Omega^n\, \mathrm{Top}_n / \mathrm{Diff}_n$, recalling that by definition $\mathrm{Top}_n / \mathrm{Diff}_n$ is the fiber of $j_m$. □




To summarise there is a bijection bewteen $\sigma$ making the lower diagram commute (the lifts) and the $\sigma'$ making the upper diagram commute.

**Theorem.**

$$\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}\, \mathrm{Top}_n / \mathrm{Diff}_n$$

***Proof.*** Looping the equivalence given by our general considerations gives us

$$\Omega\, \mathrm{Diff}(\mathbb{R}^n \text{ rel } C_\Sigma) \simeq \Omega \mathrm{Lift}(f_\tau \text{ rel } g_0)$$

and substituting our lemmas gives us that the LHS is $\mathrm{Diff}_\partial(D^n)$ and the RHS is $\Omega^{n+1}\, \mathrm{Top}_n / \mathrm{Diff}_n$. □

**Remark.** Because we are interested in the case of the disc we can assume that a smooth structure exists. In particular KS use the object $\hat{\tau}$ instead of $\tau$. This is the pullback of the tangent microbundle along some retraction of a smooth manifold onto $M$. We are therefore at liberty to take the identity for our restriction and so $\tau = \hat{\tau}$.

**Remark.** For tangent microbundles, the zero section is defined as the diagonal map into $M \times M$ which forms the total space.

**Remark.** The closure of $C$ is important because it makes the verticle maps (Kan) fibrations and hence their fibers well defined up to homotopy.

## 3 Sketching a proof of the main lemma

The real meat is in the zig zag

$$\mathrm{Diff}(M) \to \mathbf{Diff}\,\tau(M) \leftarrow \mathrm{Diff}\,\tau(M) \leftarrow \mathrm{Diff}^{\mathrm{cls}}\,\tau(M) \to \mathrm{Lift}(f_\tau)$$

and we would like to provide at least a sketch of these equivalences. [Mil64] Recall that a microbundle is a diagram

$$B \xrightarrow{i} E \xrightarrow{p} B$$

such that $pi = \mathrm{id}_B$, satisfying some local triviality condition. They satisfy a homotopy theorem, [Mil64, Thm 3.1], which says that pulling back a microbundle by homotopic maps results in isomorphic bundles Althought I cant tell if this is what KS are refering to.

**1.** This is [Classification Theorem 1]. For the first map we need to inspect the first classification theorem. For the smooth case they make use of the so called "Bundle theorem"

**Lemma.** *Let $M$ be a (metrizable) topological manifold. Let $\Sigma$ be a smooth structure on $M \times \Delta^k$ making the projection onto $\Delta^k$ a smooth submersion. Then that projection is a trivial smooth bundle over $\Delta^k$.*

In particular we see that the simplicies of $\mathrm{Diff}(M)$ consist in only trivial bundles. We need to define $\mathbf{Diff}(\tau)$, let it be the simplicial set whose $k$-simplices are germs of the smooth zero section for some $\Sigma$ a smooth structure on $\Delta^k \times \tau$, that are over $\Delta^k \times M$. Note that a smooth structure on that *bundle* is *defined* to be a smooth structure on an open neighbourhood of the zero section $U$ such that the projection $U \to M$ is a smooth submersion. If in addition the standard "inclusion" ($i$, in the diagram above) $M \to \tau(M)$ is smooth, then we call this a smooth microbundle structure.

Next they sketch a proof for the PL case and then explain how to "fix it" for the smooth case. In the PL case they show that the two functors are flexible sheaves. They then apply Gromovs theory to show that they are homotopy equivalent. The problems in the smooth case is verifying that a couple of maps are Kan fibrations. az

**2.** This is [Theorem 2.1]. Here we also need a new definition as $\mathrm{Diff}\,\tau(M)$ is not merely the set of smooth structures on the tangent microbundle. Instead it is again the simplicial set, whose k-simplicies are the set of germs of the zero section for some *smooth microbundle structure* on $\Delta^k \times \tau(M)$ over $\Delta^k \times M$. Note that there is an inclusion that goes the opposite way to our homotopy equivalence $\mathbf{Diff}\,\tau(M) \hookrightarrow \mathrm{Diff}\,\tau(M)$ This doesnt make sense to me as it seems that having a microbundle smooth structure is stronger than just having a smooth structure.... Given this the proof seems to be to take the inclusion of one of the simplicies into its microbundle

$$i : I \times \Delta^k \times M \to I \times \Delta^k \times \tau(M)$$

and then approximate this by something that is smooth in a neighbourhood of $0 \in I$ (apparently Whitney proved that this can be done) (they require that it agrees with $i$ on some subset and I think that it should also be homotopic to $i$ otherwise I dont know what they are talking about below). This defines a new microbundle and a morphisms of the old bundle to the new bundle (by the homotopy theorem, I guess we actually dont need an equivalence, only that some homotopy exists i' to i, in which case we get a bundle map the other way...?) which can be seen to be a homotopy of the old section into $\mathbf{Diff}\,\tau(M)$.

**3 + 4.** These two cases are handled together. They define some new Kan complexes $\mathscr{B}_{\mathrm{CAT}}(M)$, $\mathscr{B}^{\mathrm{cls}}_{\mathrm{CAT}}(M), \{M^m, B\,\mathrm{CAT}_m\}$, where $\mathscr{B}$ stands for bundle, that fit into a diagram

$$\begin{array}{ccccc}
\mathrm{Diff}\,\tau(M) & & \mathrm{Diff}^{\mathrm{cls}}\,\tau(M) & & \mathrm{Lift}(f_\tau) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{B}_{\mathrm{Diff}}(M) & \xleftarrow{\ \simeq\ } & \mathscr{B}^{\mathrm{cls}}_{\mathrm{Diff}}(M) & \xrightarrow{\ \simeq\ } & \{M^m, B\,\mathrm{Diff}_m\} \\
\downarrow \scriptstyle{\text{forget}} & & \downarrow \scriptstyle{\text{forget}} & & \downarrow \scriptstyle{\text{forget}} \\
\mathscr{B}_{\mathrm{Top}}(M) & \xleftarrow{\ \simeq\ } & \mathscr{B}^{\mathrm{cls}}_{\mathrm{Top}}(M) & \xrightarrow{\ \simeq\ } & \{M^m, B\,\mathrm{Top}_m\}
\end{array}$$

Where the spaces that we are interested in are the fibers of the verticle maps (which are Kan fibrations), and hence we get a homotopy equivalence between the fibers. Note that $\mathrm{Diff}^{\mathrm{cls}}$ is *defined* to be this fiber, while the others we need to show it.

So first what are the spaces. The easiest to define is $\{X, Y\}$ which is just the singular complex of $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$. $\mathscr{B}^{\mathrm{cls}}_{\mathrm{CAT}}(M)$ has $k$-simplicies given by CAT m-microbundles $\gamma$ over $\Delta^k \times M$ equiped with a CAT morphism $g : \gamma \to \gamma^m_{\mathrm{CAT}}$, we further identify these microbundles with classifying maps if their germs of the zero section agree; thus it is microbundles with a map to the classifying space locally around the zero section. $\mathscr{B}_{\mathrm{CAT}}(M)$ is the same but without the map to the classifying space.

Next the equivalences are the forgetful maps [KS16, Assertion 2.4]. They more or less say that their equivalence status is obvious.... I dont know how to "see" equivalences like this for simplicial sets.

## References


[BL] Dan Burghelea and Richard Lashof. The homotopy type of the space of diffeomorphisms. I. 196(0):1–36.

[KS16] Robion C. Kirby and Laurence C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations: AM-88*. Number 88 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2016.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.

[Mil64] J. Milnor. Microbundles: Part I. *Topology*, 3:53–80, July 1964.